EVALUATING
ACADEMIC READINESS
FOR APPRENTICESHIP TRAINING
Revised for
ACCESS TO APPRENTICESHIP

MATHEMATICS SKILL
PROPERTIES OF TRIANGLES

AN ACADEMIC SKILLS MANUAL
for
The Precision Machining And Tooling Trades

This trade group includes the following trades:
General Machinist, Tool & Die Maker,
Mould Maker, Pattern Maker, and
Machine-Tool Builder Integrator

Workplace Support Services Branch
Ontario Ministry of Training, Colleges and Universities

Revised 2011

In preparing these Academic Skills Manuals we have used passages, diagrams and questions similar to those an apprentice might find in a text, guide or trade manual.

This trade related material is not intended to instruct you in your trade. It is used only to demonstrate how understanding an academic skill will help you find and use the information you need.

MATHEMATICS SKILL
PROPERTIES OF TRIANGLES

An academic skill required for the study of the Precision Machining and Tooling Trades

INTRODUCTION

*A **triangle** is a three-sided, closed figure formed from three connected, straight lines*. In the machining trades, you will find that triangles are one of the basic shapes that will appear in your work again and again. You will find them when working with blueprints and layout procedures, and in fabrication. Whenever you bend a metal bar and connect the two outside ends, you form a triangle.

Triangles have some special properties that can be used to figure out unknown measurements. But first we will look at some of the basic properties of triangles.
This skills manual will look at:

- Angles
- How a triangle is defined
- The types of triangles
  - according to the length of the sides
  - according to their angles
- Pythagoras' Theorem, including
  - using the theorem to measure indirectly
- Congruent and similar triangles
- Finding unknown angles in a triangle
- Calculating the perimeter of a triangle
- Calculating the area of a triangle
- Calculating the volume of a three dimensional triangular solid

At the end of the skills manual you can test yourself to see how well you understand the information presented.

ANGLES

Every triangle has three sides and forms three angles at its corners. We start by looking at angles.

*An **angle** is formed by two lines joined at a common point called an endpoint or a **vertex.*** Angles can be named either by using the word angle in the name, or by using the symbol for angle, $\angle$.

The angle in Figure 1 could be named Angle ABC, or ∠ABC, or Angle B, or ∠ B.

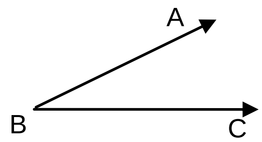


FIGURE 1: Naming an Angle
An angle is named by its letters. The letter at the vertex is always the middle letter of the angle name. Angle ABC can also be called angle B, or ∠ B

Measuring angles
An arc is a part of a circle, and a circle is measured in degrees. A curved arc can be drawn between the two lines which make an angle. (See Figure 2.)

- The center of the arc is the vertex (point where the two lines meet) of the angle.

To measure the size of an angle, we measure the number of degrees in the arc drawn between the two lines forming the angle.

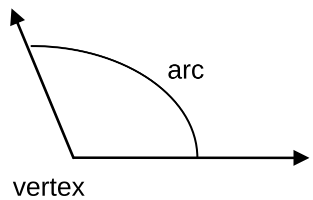


FIGURE 2 Measuring An Angle
To measure an angle, measure the number of degrees in the arc.

Like a circle, angles are measured in units called **degrees (^{o})**.

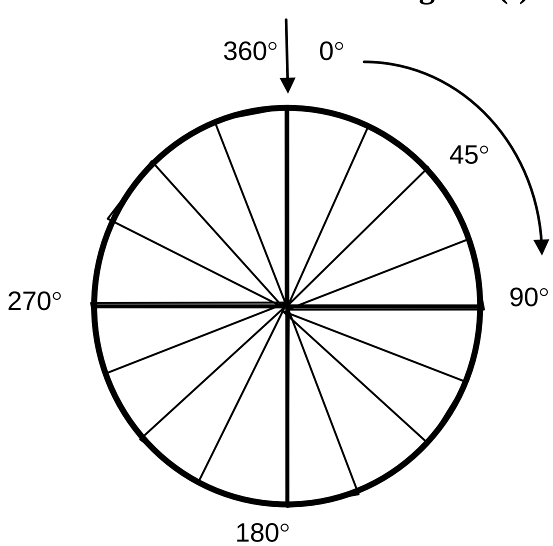


FIGURE 3: Degrees in a Circle
Some of the angles formed by dividing a circle into 360°

To get a picture of the size of one degree, think of the outside of a circle divided into 360 equal parts. Use Figure 3 as you read on to see the relationship between a circle and angles:

- Lines drawn from the centre of the circle to each of the 360 dividing marks make 360 equal angles, each measuring 1 degree.
- One complete rotation around the outside of a circle measures 360°. (See Figure 3.)
- One quarter of a circle measures 90°.
- One half of a circle measures 180°.
- Notice that an angle of 180° is the same as a straight line.
- Three quarters of a circle measures 270°.

We measure angles using a *protractor*. A protractor is calibrated to measure the number of degrees in the arc, so we don't actually have to draw the arc to measure an angle.

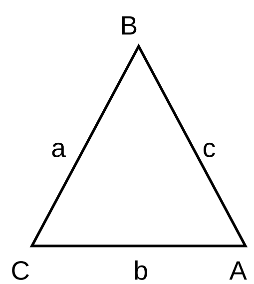
Because we often have to know the number of degrees between two connected lines, it is important to be able to measure angles. Sometimes, though, we can't measure an angle directly, so we have to find its value indirectly.

Any angle can be changed into a triangle by drawing a straight line to connect the other two lines forming the triangle. Certain properties of triangles enable us to find angles indirectly.

DEFINITION OF A TRIANGLE

A ***triangle*** *is a three-sided, closed figure formed by three connected straight lines.*
(See Figure 4.)

- ♦ Where two sides of a triangle meet, an angle is formed.
- ♦ Every triangle has three angles and three sides.
- ♦ A triangle is named using the three letters that name its angles:
  - We use the word *triangle* (triangle ABC) or
  - we use Δ, the symbol for triangle (Δ ABC).
- ♦ ***The sum of the three angles in every triangle adds up to 180°.***
  - This important fact about triangles is useful in doing calculations



Triangle ABC
Also called ΔABC or Δabc

FIGURE 4: Naming a Triangle: A triangle is usually named by the three letters that name either its three different angles. The symbol Δ or the word "triangle" is used in naming triangles.

In Figure 4, the angles are named by capital letters and the sides are named by lowercase letters.

- Commonly, the letter used to name a side is the same as the capital letter of the angle that is opposite that side.

Example: In Figure 4, ***angle A*** is in the lower right corner and ***side a*** is opposite that angle on the left side.

FINDING UNKNOWN ANGLES

The sum of the three angles in any triangle equals 180°. Therefore, if we know the size of two angles in a triangle, we can find the size of the third.

To find an unknown angle when two of the angles are known:

1. Add the value of the two known angles.
2. Subtract the addition answer from 180 .
3. The subtraction answer is the value of the third angle.

Example: If one angle in a triangle is 40° and the second angle is 85°, what is the value of the third angle?

40° + 85° = 125°	Add the two known angles together.
180° − 125° = 55°	Subtract this answer from 180° to get the value of the third angle.

The third angle measures 55°.

TYPES OF TRIANGLES

Triangles can be classified in several ways. They are classified by the relationship between the lengths of the three sides. They are also classified by the types of angles they have.

Types of Triangles According to the Lengths of the Sides

The types of triangles are named according to the length of their sides. See Figure 5.

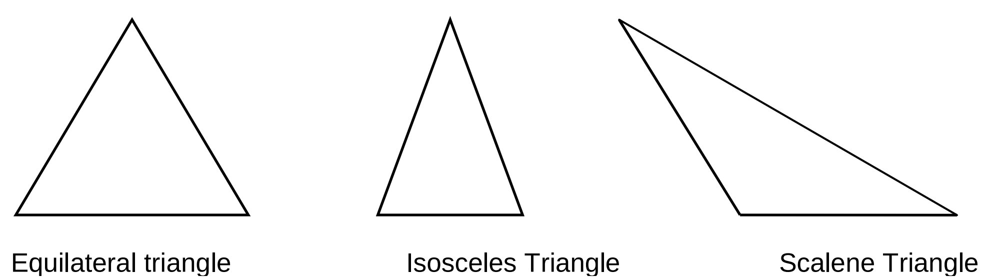


FIGURE 5: Types of triangles
In isosceles and scalene triangles, the largest angle is opposite the longest side. The smallest angle is opposite the smallest angle.

1. ***Equilateral triangles***
   - The sides of an equilateral; triangle are equal in length.
   - The angles are also equal.
   - Since the sum of the angles in a triangle is 180°, each angle in an equilateral triangle is 60°. (180° ÷ 3 = 60°)

2. ***Isosceles triangles***
   - Two sides of an isosceles triangle are equal in length.
   - The two angles opposite the equal sides are also equal.
   - The unequal side is called the base

3. ***Scalene triangles***
   - The scalene triangle has no equal sides.
   - It has no equal angles.

   In isosceles and scalene triangles, the largest angle is opposite the longest side. The smallest angle is opposite the shortest side.

Types of Triangles According to the Angles Triangles can also be classified by the types of angles they have. Angles are named according degree of angle. Here are the names of some important angles:

- A ***right angle*** is 90° .
- An ***acute angle*** is greater than 0° but less than 90° .
- An ***obtuse angle*** is greater than 90° but less than 180 º .

We know that in every triangle, the sum of the three angles is 180°. Every triangle has at least two acute angles (less than 90°) The third angle can be a right angle, an acute angle or an obtuse angle.

Triangles can be named according to the type of third angle they have. See Figure 6.

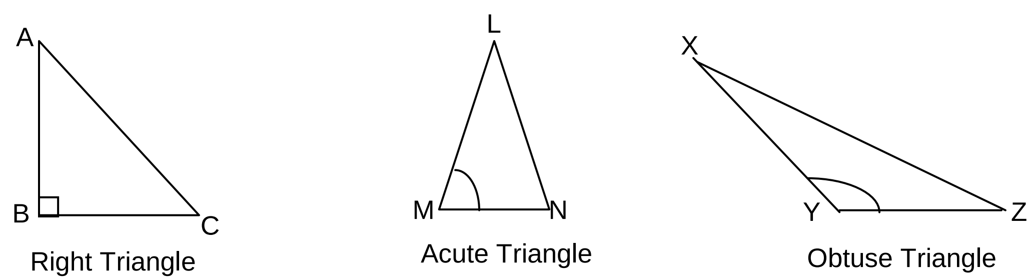


FIGURE 6:Triangles Named According to Their Angles

- A ***right triangle*** has one right angle and two acute angles.
- An ***acute triangle*** has three acute angles.
- An ***obtuse triangle*** has one obtuse angle and two acute angles.

The Right Triangle
A right triangle has some special relationships between its angles and sides. This makes it possible to calculate unknown angles or lengths.

Because the right triangle is the triangle most often used in doing calculations, it is important to know the parts of a right triangle. Examine Figure 7 and compare the triangle with the description that follows.

Figure 7 is a typical right triangle. Notice the square box at angle C in the diagram; it is the symbol for a right angle of 90^{o}.

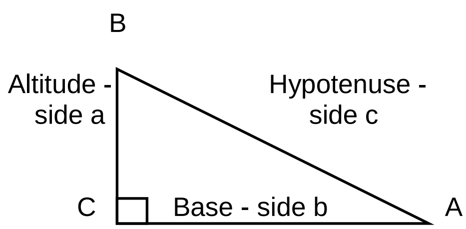


FIGURE 7: Parts of a Right Triangle

In Right Angle Triangle ABC:

1. The ***hypotenuse*** is the side opposite the right angle (side c).
   - The hypotenuse is always the longest side.
2. The ***base*** is the side the triangle rests on (side b).
3. The ***altitude*** or ***height*** is the vertical side (side a).

PYTHAGORAS' THEOREM

You can find the length of an unknown side of a right triangle by using Pythagoras' theorem. You can use this method if you know the lengths of two sides of any right angle triangle.

Continue to refer to Figure 7 as we examine Pythagoras' theorem.

Pythagoras' theorem states that, in a right triangle:
The square of the hypotenuse is equal to the sum of the squares of the other two sides. sides.

The formula expressing this theorem is:

for $\triangle$ ABC : $c^2 = a^2 + b^2$

where *c* is the hypotenuse, *a* is the altitude and *b* is the base

If the lengths of two sides of a right triangle are known, the length of the third side can be found using this formula. To find *c*, the formula is written as:

$$c^2 = a^2 + b^2$$

To find the answer, we have to find the square root.

$$c = \sqrt{a^2 + b^2}$$

To find *a*, the formula is rearranged and written as:

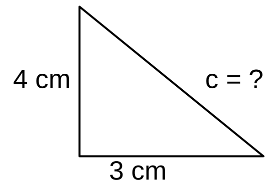
$$a^2 = c^2 - b^2$$

To find *b*, the formula is written as:

$$b^2 = c^2 - a^2$$

Example: Find the hypotenuse of a right triangle if the altitude is 4 cm and the base is 3 cm.

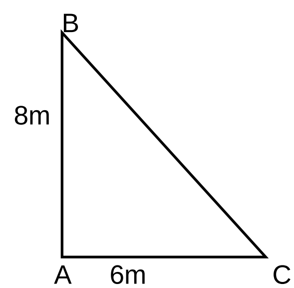
Known:
a = 4 cm
b = 3 cm
c = ?



$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2 + b^2}$$
$$c = \sqrt{4^2 + 3^2}$$
$$c = \sqrt{16 + 9}$$
$$c = \sqrt{25}$$
$$c = 5\text{cm}$$

Example: Find the length of BC if AB equals 8 m and AC equals 6 m.

Look at the diagram. Side BC is opposite the right angle and must be the hypotenuse or *c* in the formula, so:



$$c^2 = a^2 + b^2$$
$$c = \sqrt{8^2 + 6^2}$$
$$c + \sqrt{64 + 36}$$
$$c = \sqrt{100}$$
$$c = 10 \text{ m}$$

Sometimes the square root does not have a perfect square: If the square root doesn't come out evenly, the answer can be written with the square root sign as in the example below.

Example: Find the length of *a* in a right triangle ABC if *c* is 12 m and *b* equals 9 m.

$$a^2 = c^2 - b^2$$
$$a = \sqrt{c^2 - b^2}$$
$$a = \sqrt{12^2 - 9^2}$$
$$a = \sqrt{144 + 81}$$
$$a = \sqrt{235}\text{ m}$$

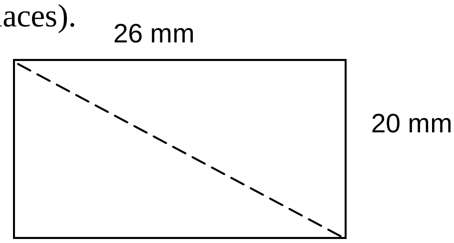
This can be an acceptable answer. But, if you are using Pythagoras' theorem to find a measurement you will be using later, you will have to find its square root. It is a good idea to have a calculator handy to give you the square root.

We use Pythagoras' theorem to make sure a rectangle is a true rectangle, with right angles at the corners.

- First, we calculate the theoretical length of the diagonal using Pythagoras' theorem.
- Then we measure the diagonal of the rectangle we are testing.
- The measured length of the diagonal is compared to the calculated length.
- If the lengths are the same the rectangle is true.

Example: A rectangular block has a length of 26 mm and a width of 20 mm. Find the diagonal measure (to two decimal places).

$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2 + b^2}$$
$$c = \sqrt{26^2 + 20^2}$$
$$c = \sqrt{676 + 400}$$
$$c = \sqrt{1076}$$



Use a calculator to find the square root. Key in 1076 and hit the square root symbol to get the answer.

$$c = 32.802439$$

The diagonal should measure 32.80 mm if the block is a true rectangle.

Using Pythagoras's Theorem to Measure Indirectly

We may need to find the length of something that we can't measure directly. Pythagoras' Theorem can help solve questions that involve a right triangle.

Note: *It is always a good idea to draw a diagram first to help you visualize a problem.*

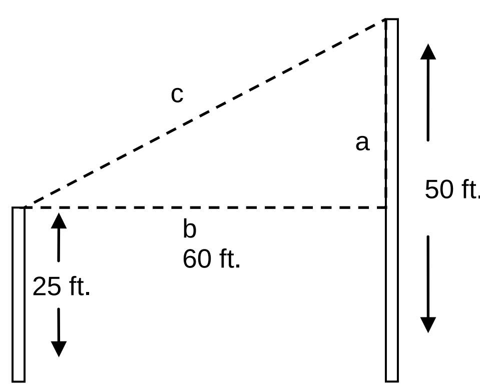
Example: Two poles, one 50 ft and the other 25 ft high, are 60 ft apart. Find the distance from the top of one pole to the top of the other pole.

First, draw a diagram to show that that b equals 60 ft. (We have done one for you.)

List what you know from the problem:

- Side *a* is the difference between the length of the two poles.
- a = 25 ft (50 ft - 25 ft)
- b = 60 ft
- c = the hypotenuse.

Put what you know into Pythagoras' theorem.

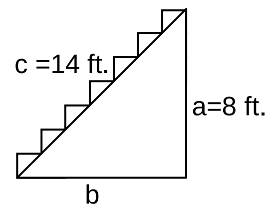


$$c = \sqrt{a^2 + b^2}$$

$c = \sqrt{25^2 + 60^2}$
$c = \sqrt{225 + 3600}$ Use a calculator to find the square root.
$c = \sqrt{3825}$
$c = 61.847$ ft.

Example: The second floor of a shop is 8 feet above the first floor. The diagonal length of the staircase is 14 feet. What is the run of the stairs to one decimal place? In other words, what is the measurement of the base of the stairs?

Draw a diagram and mark the sides 'a', 'b', and 'c'.



List what you know from the problem:

- c (the hypotenuse) = 14 ft
- a (the height) = 8 ft.
- b = the base of the stairs

Put what you know into Pythagoras' theorem.

$c^2 = a^2 + b^2$
$b^2 = c^2 - a^2$
$b = \sqrt{c^2 - a^2}$
$b = \sqrt{196 - 64}$
$b = \sqrt{132}$
$b = 11.5$ ft The base of the stairs will be 11.5 ft.

Using Pythagoras' Theorem to Solve Other Figures

You can make right triangles from an isosceles triangle, a square or a rectangle. Look at Figure 8.

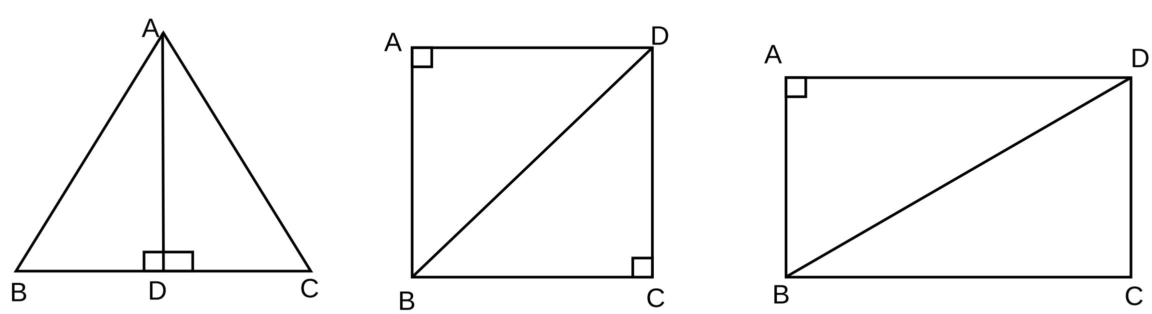


FIGURE 8: Bisect an isosceles triangle, a square, or a rectangle to form right angle triangles.

ΔABC forms two right angle triangles ADB and ADC.
Square ABCD forms right triangles ABD and DCB;
Rectangle ABCD forms right triangles ABD and DBC

Figure 8 shows that:

- Two identical right triangle are made if we bisect (cut in half) the base using a line perpendicular (at right angles) to the base.

- ♦ Two identical right triangles are formed if we divide a square or rectangle with a diagonal line between two of its corners.

Note: *If a parallelogram is divided in two by a diagonal line, two congruent triangles are formed but they are not right triangles.*

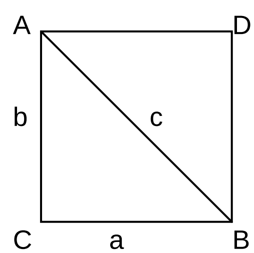
Using Pythagoras' Theorem to Solve a Square

In a square, if we know the length of the diagonal line between any opposite corners, you can use Pythagoras' Theorem to find the length of any side.

- The length of every side in a square is the same; and
- The diagonal forms a pair of right triangles in the square.

Example: Find the length of the sides of the square below if the length of the diagonal is 24cm.

A square divided by a diagonal forms a right triangle. The diagonal, BC, is the hypotenuse. CB is the same as side a of the right triangle. AC is the same as side b of the triangle.



Known:

AB = CB because ABCD is a square
side a = side b
side c (the hypotenuse) = 24cm

Since a and b are equal, we can replace b^2 with another a^2 in the formula.

$c^2 = a^2 + b^2$

$c^2 = a^2 + a^2$ replace b^2 with a^2

$c^2 = 2a^2$

$24^2 = 2a^2$ replace c with 24

$$a^2 = \frac{24^2}{2}$$

$$a^2 = \frac{576}{2}$$ divide both sides by 2

$a^2 = 288$

$\sqrt{a^2} = \sqrt{288}$

$a = 16.97 \ldots$

The sides of the square are 16.97...cm long. Round the answer off to 17 cm.

CONGRUENT AND SIMILAR TRIANGLES

Triangles that are exactly the same size and shape are called ***congruent triangles***. In congruent triangles:

- all corresponding angles, and
- the lengths of all corresponding sides are identical. (See Fig 9.)

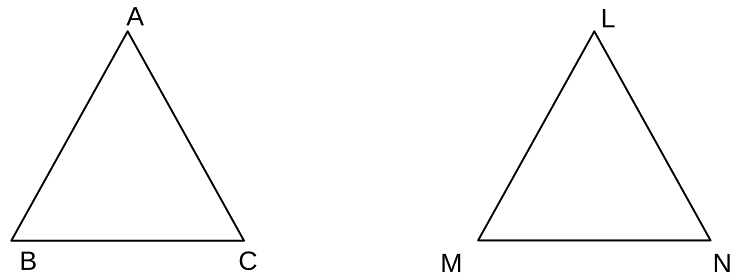


FIGURE 9: TRIANGLE ABC AND TRIANGLE LMN ARE CONGRUENT.

Sides are all equal	Angles are all equal
AB = LM	∠A = ∠L
BC = MN	∠B = ∠M
CA = NL	∠C =∠N

Similar triangles have the same shape but they differ in size. Look at Figure 10.

- Because the shape of a triangle is determined by its angles, triangles with the same shape have the same corresponding angles.
  - In other words, *if the triangles are in the same position*:
    - The angle in the lower right corner of two similar triangles will be the same; and
    - The angles in the other two corners will also be the same.
- The sides of similar triangles are not equal.

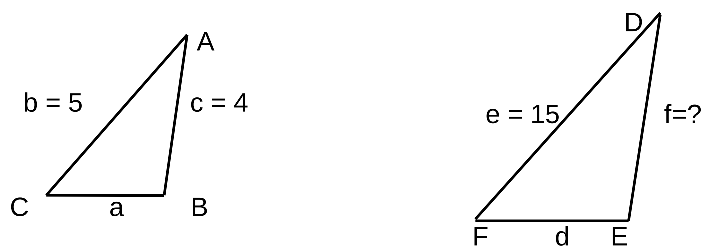


FIGURE 10: TRIANGLE ABC AND TRIANGLE DEF ARE SIMILAR.

Corresponding sides are not equal:	Corresponding angles are equal:
$b \neq c$	∠A = ∠D,
$c \neq f$	∠B =∠E
$a \neq d$	∠C =∠F

Although the three corresponding angles are the same in a pair of similar triangles, the lengths of the corresponding sides are different, as shown in Figure 10. However:

- ♦ *In similar triangles, the **ratios** of the corresponding sides are equal.*

Let's look at what this means.

- In any triangle ABC, the ratio of sides a and b is formed by putting the length of side a over the length of side b.

$$\frac{\text{side a}}{\text{side b}}$$

- In Figure 10, triangle DEF is similar to triangle ABC. Side d and side e are in the same position as side a and b. The ratio

$$\frac{\text{side d}}{\text{side e}}$$

is equal to the ratio of side a over side b.

- Since the two ratios are equal, they could be made into a proportion by writing them with an equal sign between them.

$$\frac{\text{side a}}{\text{side b}} = \frac{\text{side d}}{\text{side e}}$$

When the length of three of the sides is known this equation can be used to find the fourth, unknown side.

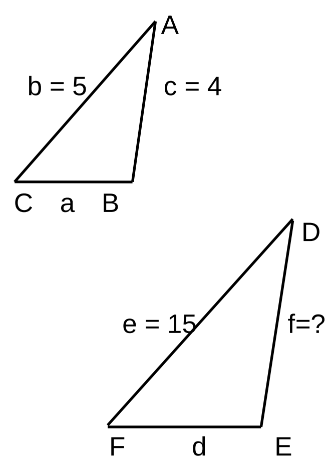
Example: Use the triangles in Figure 10 to see how to find an unknown side of a similar triangle.

Draw and label the triangles.

Known:

1. The two triangles in the diagram are similar.
2. The ratios of corresponding sides are equal.
3. c = 4
4. b = 5
5. e = 15
6. The ratio of side c to side b in triangle ABC = the ratio of side f to side e in triangle DEF.

$$\frac{\text{side c}}{\text{side b}} = \frac{\text{side f}}{\text{side e}}$$



Substitute the known values into the equation

$\frac{4}{5} = \frac{f}{15}$

$\frac{f}{15} = \frac{4}{5}$ reverse the equation and multiply by 15 to isolate f

f = $\frac{4 \times 15}{5}$ do the calculation to solve for f

f = 12

The length of side f is 12.

PERIMETER OF A TRIANGLE

The perimeter of a triangle is the distance around the outside. It equals the sum of the lengths of its sides.

To find the perimeter of a triangle, add the lengths of the three sides together.

- Remember, the lengths must be in the same units; you can't add centimeters and meters together.
- Change either the centimeters to meters or meters to centimeters so all quantities have the same units.

The formula for finding perimeter is:

P = a + b + c

Example: Find the perimeter of a triangle in which side a measures 6 cm, side b measures 10 cm, and side c measures 9 cm.

Known
a = 6
b = 10
c = 9

P = a + b + c	Substitute known vales
P = 6 cm + 10 cm + 9 cm	Solve for P
P = 25 cm	

AREA OF A TRIANGLE

The area of a triangle is the surface contained within its boundaries. It is equal to one-half the base multiplied by the altitude or height. The ***altitude or height*** of a triangle is found by measuring a line which is perpendicular to the base and extends to the point where the other two sides meet. Look at Figure 11.

The formula to find the area of a triangle is:

Area = ½ of the **b**ase times the **h**eight (or altitude).

A= ½ bh

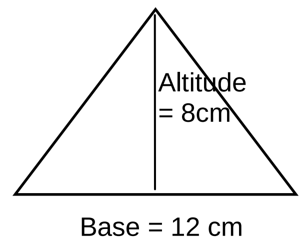


FIGURE 11: Use The Lengths Of The Base And The Height Of A Triangle To Find Its Area.

Example: Find the area of the triangle in Figure 11. The triangle has a base of 12 cm and an altitude of 8 cm.

Remember: *The units of length must be the same. When two identical units are multiplied, they become squared units in the answer.*

Known
b= 12 cm
h = 8 cm

A = ½ *bh*	*write the formula and substitute known quantities*
A = ½ (12 cm)(8cm)	*solve for A*
= ½ (96 cm^2)	units are squared
= 48 cm^2	

VOLUME OF A THREE DIMENSIONAL TRIANGULAR OBJECT

To find the volume (V) of a solid triangular object:

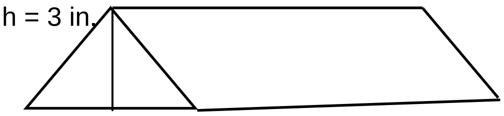
1. Multiply the length (*l*) by the area (*A*) of the triangle.
   - The area of the triangle equals 1/2 base times height.

The formula for finding the volume of a triangular solid is:

V = *1/2 bhl* where b is the base, l is the length , and h is the height of the figure
or,
V = *lA* where l is the length of the figure and A is the area of the triangular face.

Remember: *The units of measure must be the same. When three identical units are multiplied together they become cubed, or cubic units.*

Example: Find the volume of the solid triangular object below.



You can solve this problem by using either one of the formulae.

Known:
b = 2 in
h = 3 in
l= 7 in

V = ½ bhl
V = ½(2in x 3in x 7in)
V= ½ (42in^3)
V = 21 in^3

Or

V = lA
find A
A = ½ bh
= ½(2" x 3")
= 3 in^2

V = l(Area)
= 7in(3in^2)
= 21 cu in
or, = 21 in^3

Remember: *The units of measure must be the same. When three identical units are multiplied together they become cubed, or cubic units.*

The following questions involve properties of triangles. Answers are on the last page.

1. The sum of the angles in a triangle add up to _____________ .

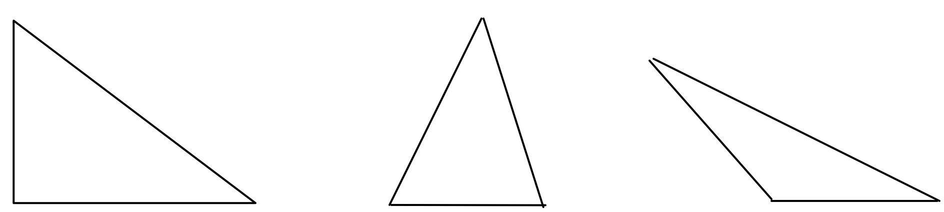
2. When all three sides of a triangle are equal in length, the triangle is called an _____________ triangle.

3. The angles in an equilateral triangle are also equal. Each angle in an equilateral triangle measures _____________ .

4. An acute angle is greater than 0^o but less than ______ ^{o}. An acute triangle has ____________ acute angles.

5. An obtuse angle is greater than 90^o but less than _____ ^{o}. An obtuse triangle contains two acute angles and one ______________ angle.

6.Label the following triangles as acute, obtuse and right:



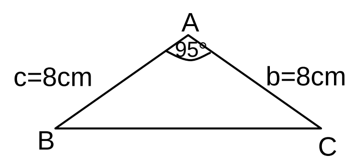
A. ____________ B. ___________ C. ____________

7. A right triangle has one right angle that measures _____________ . The other two angles are acute angles.

8. If one angle in a triangle is 65° and the second angle is 75°, what is the value of the third angle? __________________

9. In an isosceles triangle ABC, side b = side c and angle B = angle C. Angle A equals 95°.

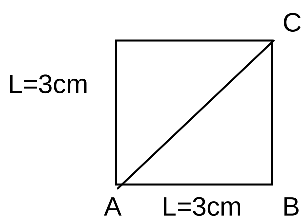
   What is the size of angle B?
   What is the size of angle C?



10. What is the perimeter of a triangle with sides 14 cm, 32 cm and .5 *meters* (first make sure all your units are the same)? _____________

11. What is the area of a triangle with a base of 10 cm and a height of 4 cm? ____________

12. What is the area of the triangle ABC in the diagram?

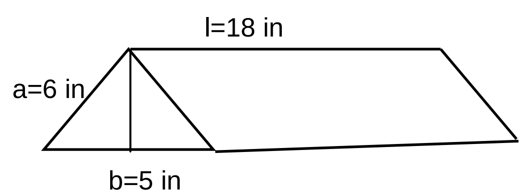


Since the base and the height of the triangle are known, you can find the area.

13. If a triangle with two equal sides is bisected by a line ___________ to the base, two

congruent triangles are formed.

14. What is the volume of the solid triangular figure below? _____________________



15. In two ***similar*** triangles ABC and DEF, side a measures 4 ft and corresponding side d measures 8 ft. Side b measures 12 ft and the measurement of its corresponding side e is unknown. Find the length of side e, using the proportion:

$$\frac{\text{side a}}{\text{side b}} = \frac{\text{side d}}{\text{sice e}}$$

ANSWER PAGE

1. 180

2. equilateral

3. 60

4. 90 , three

5. 180 , obtuse

6. A. right, B. acute, C. obtuse

7. 90

8. 65 + 75 = 140
$180^o - 140^o = 40^o$
The third angle is 40^o

9. 180 − 95 = 85 Since angle B and C are equal, divide 85 by 2 to get the measurement of both angles.
$85^o \div 2 = 42.5^o$
Angle B = 42.5^o
Angle C = 42.5^o

10. First change the .5 m to cm before you add. .5 m = 50 cm
14 cm + 32 cm + 50 cm = 96 cm

11. A = ½ bh
= ½ 10 cm (4 cm)
= ½ 40 cm^2
= 20 cm^2

12. A = ½ bh
= ½ 3 cm (3 cm)
= ½ (9cm^2)
= 4.5 cm^2

13.perpendicular

14. Either

V = ½ bhl
= ½ (6in x 5in x 18in)
= ½ ($540in^3$)
= $270in^3$

or

V = lA
A = ½ (6 in) (5 in) = 15 sq in
V = l (Area)
= 18 in (15 sq in)
= 270 cu in

15.

$$\frac{\text{side a}}{\text{side b}} = \frac{\text{side d}}{\text{sice e}}$$

$$\frac{4}{12} = \frac{8}{e}$$

Cross-multiply: 4(side e) = 96
Divide by 4: side e = 24 ft